3

Growth & Decay II

3.1 Growth and Decay

We can solve any exponential equation using logarithms.

> A population that is growing at a constant rate will have
>
> $$P(\text{t}) = P(0)\, e^{rt}$$
>
> members after time t, where
>
> - $P(0)$ is the initial population and
> - r is the constant growth rate per unit time.

Example

The population of China was 850, 000, 000 in 1990 and was growing at the rate of 4% per year.

When did the population reach 1,000,000,000?

Answer

The initial population (in 1990) is $P(0)$ = 850, 000, 000.

Rewrite 4% as a decimal number.

The growth rate is 0.04 per year.

The model is $P(t) = P(0)e^{0.04t}$, we need to find when $P(t)$ = 1,000,000,000

$$\text{Put} \quad 850{,}000{,}000e^{0.04t} = 1{,}000{,}000{,}000$$

$$e^{0.04t} = \frac{1{,}000{,}000{,}000}{850{,}000{,}000}$$

Rewrite as an exponential equation.

$$= \frac{100}{85}$$

$$0.04t = \ln\left(\frac{100}{85}\right)$$

Take logarithms of both sides.

$$t = \frac{\ln\left(\frac{100}{85}\right)}{0.04}$$

$$= 4.1 \ \ (2 \text{ sf})$$

The population reached 1,000,000,000 in 1994.

Example

The population of China was 850, 000, 000 in 1990 and reached 1,000,000,000 in 1994. If it grew at a constant growth rate, what is this growth rate?

Answer

The initial population (in 1990) is $P(0)$ = 850, 000, 000.

The population in 1994 is $P(4)$ = 1,000,000,000

The growth model is $P(t) = P(0)e^{rt}$, we need to find the rate r.

$$\text{Put} \quad 850,000,000e^{r\times 4} = 1,000,000,000$$

$$e^{4r} = \frac{1,000,000,000}{850,000,000}$$

$$= \frac{100}{85}$$

$$4r = \ln\left(\frac{100}{85}\right)$$

$$r = \frac{\ln\left(\frac{100}{85}\right)}{4}$$

$$= 0.041 \text{ (2 sf)}$$

The population grew at a constant rate of 4.1% per year.

Problems 3.1

1. The population of the earth is now 4 billion, and is increasing at a constant rate of 2% per year. If it continues to grow at this rate, when will the population reach 5 billion?

2. The population in Britain in 1600 is believed to have been about 5 million. Three hundred fifty years later the population had increased to 50 million. What was the average percentage growth during that period? (Assume that the growth is constant.)

3. Radioactive radium decays at a rate of 0.044% per year. How many years does it take 10 gm of radium to decay so that only 8 gm of radium remains? How long will it take for a further 2 gm of radium to decay?

4. The ratio of radioactive isotope C^{14} to the regular isotope C^{12} of carbon is fixed in the atmosphere. Living matter breathes in air, and this same ratio of C^{14} to C^{12} is found in all its cells. When it dies and can no take breath in air, the amount of C^{14} begins to decay at a constant rate of 1.24 $\times 10^{-4}$. This is the principle of carbon dating.
If the amount $Q(t)$ of radioactive carbon C^{14} in a human bone is measured to be 58% of the amount $Q(0)$ found in the atmosphere, how old is the bone?

3.2 Doubling Time and Half-life.

The growth rate of a population is usually quite small, and its hard to imagine how fast a polulation is actually growing. Because of this the *doubling time* is often quoted instead. The doubling time of a population is the time it takes to double.

Example

If a town had an initial population of 1000 and a doubling time of 30 years, then the population would be 2000 after 30 years, 4000 after another 30 years, 8000 after a further 30 years (ie. after a total of 90 years from the beginning).

A population growing at a constant growth rate r will double in size every $\frac{\ln 2}{r}$ units of time

Reason

The population growth model is $P(t) = P(0)e^{rt}$, where $P(0)$ is the starting time.
If the population doubles, then $P(t) = 2P(0)$ and we have the equation,

$$
\begin{aligned}
2P(0) &= P(0)e^{rt} \\
e^{rt} &= 2 \\
rt &= \ln 2 \\
t &= \frac{\ln 2}{r}
\end{aligned}
$$

so the population doubles after $\frac{\ln 2}{r}$ units of time.

Example

If a town had an initial population of 1000 and grew at a constant rate, if the population doubled every 30 years, what is the growth rate?

Answer

$$\text{Put } \frac{\ln 2}{r} = 30,$$

$$\text{then } \ln 2 = 30r$$

$$r = \frac{\ln 2}{30} = 0.023$$

Similarly, the decay rate of a quantity is usually quite small, and its hard to imagine how fast it decays. Because of this the *half-life* is often quoted instead. The *half-life* of a quantity is the time it takes to halve.

> A quanity which is decaying at a constant rate will have the amount
>
> $$Q(t) = Q(0)\, e^{-rt}$$
>
> left after time t, where
>
> - $Q(0)$ is the initial amount and
> - r is the constant *decay* rate per unit time.

Example

One kilogram of a radioactive isotope of iodine has a half life of 7.967 years. After this period of time only 500 gm will remain. After a further 7.967 years only 250 gm (ie. half of 500gm) will remain.

> A population decaying at a constant decay rate r will be reduced by half every $\frac{\ln 2}{r}$ units of time

Reason

The decay model is $Q(t) = Q(0)e^{-rt}$, where $Q(0)$ is the initial amount.

If the quantity halves, then $Q(t) = 0.5Q(0)$ and we have:

$$0.5Q(0) = Q(0)e^{-rt}$$

$$0.5 = e^{-rt}$$

$$e^{rt} = 2$$

$$rt = \ln 2$$

$$t = \frac{\ln 2}{r}$$

Example

One kilogram of a radioactive isotope of iodine decays at a rate of 8.7% per day. What is its half life

Answer

$$t = \frac{\ln 2}{r},$$

$$\begin{aligned} t &= \frac{\ln 2}{0.087} \\ &= 7.967 \text{ years} \end{aligned}$$

The half life is 7.967 years.

Problems 3.2

1. A culture of bacteria doubles in weight every 24 hours. If it originally weighed 10 g, what would be its weight after 18 hours?

2. The half-life of radium is 1590 years. If 10 g of radium is left after 1000 years, how much was there originally?

A

Appendix: Answers

Section 1.1

1(a) $\log_2 16 = 4$ (b) $\log_2 1024 = 10$ (c) $\log_2 0.5 = -1$ (d) $\log_2 1 = 0$

(e) $\log_3 81 = 4$ (f) $\log_4 1024 = 5$ (g) $\log_4 0.5 = -0.5$ (h) $\log_{10} 1 = 0$

2(a) 2 (b) 4 (c) 0 (d) –1

(e) 1 (f) 2 (g) 0 (h) –2

Section 1.2

1 (a) 0 (b) 0.6931 (c) 1.099 (d) 1.386 (e) 2.197

(f) 0 (g) 0.301 (h) 0.6021 (i) –1.386 (j) 0.5493

2 (a) 0.6931 (b) 0.9163 (c) 0.6931 (d) –1.099

(e) 0 (f) 0.1505 (g) 0.3010 (h) –0.2386

3 (a) 0.5669 (b) 0.3838 (c) 0.3973 (d) 0.2303

4 (a) 9.70 billion (b) 41.81 years

Section 2.2

1 (a) ln 18 (b) ln 8 (c) ln 25 (d) ln 20

(e) ln 25 (f) ln 0.5 (g) $\ln 3e^2$ (h) ln 1

2(a) 2 (b) 3 Note: $x = -2$ is not a solution as only positive numbers are in the domain of ln x.

3(a) 2.036 (b) 2.010 (c) 2.983 (d) 0.5350

Section 3.1

1. 11.2 years
2. 0.6579%
3. 507.1 years, 653.8 years
4. 4392.7 years

Section 3.2

1. 16.82 g
2. 15.47 g